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TWO-PHASE STEFAN PROBLEMS IN NON-CYLINDRICAL DOMAINS

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Abstract. In this paper we discuss a two-phase Stefan problem in a non-cylindrical (time-dependent) domain. This work is motivated by the phase change arising in the Czochralski crystal growth process. The time-dependence of domain is a mathematical description of the situation in which the material domain changes its shape with time by the crystal growth. We consider the so-called enthalpy formulation for it and give its solvability, assuming that the time-dependence of the material domain is prescribed and smooth enough in time. Our main idea is to apply the theory of quasi-linear equations of parabolic type.

1. Introduction

Czochralski pulling method is widely used for the production of a column of single silicon crystal from the melt. The idea of pulling method due to Czochralski is quite simple. A crucible, equipped with heating system, contains the melt substance and a pul-rod with seed crystal, which moves vertically and rotates flexibly, is positioned above the crucible (see Fig.1). The rod is dipped into the melt, and then lifted slowly with an appropriate speed v_p so that a meniscus surface is formed below the seed crystal and the melt attached to the crystal solidifies continuously. By controlling some thermal situations in the process one obtains the growth of a single crystal column with a desired radius as well as a desired growth pattern of the solid-liquid interface and temperature pattern in the crystal in order to improve the crystal quality.

In such a model of crystal growth the shape of crystal is determined by three kinetic equations of three interfaces between crystal-melt, melt-gas and gas-crystal. But, in this paper we suppose that the crystal radius is controlled to be constant and the trijunction curve on which three interfaces meet is prescribed, too. This might be designed by a good choice of the pulling velocity. As a consequence we may assume that the movement of the material domain is prescribed.

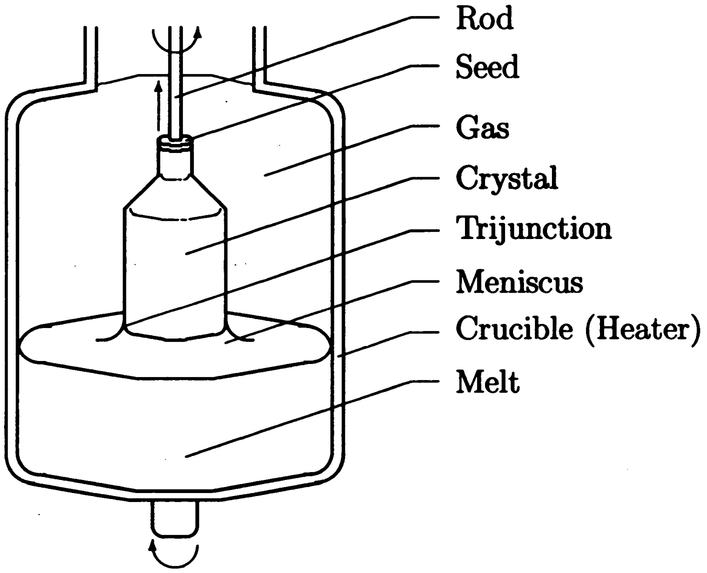

Fig.1

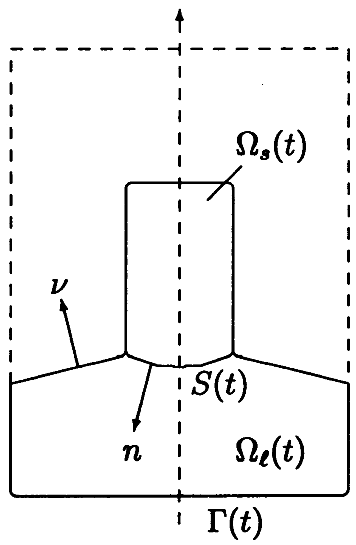

Fig.2

We use the following notation (see Fig.2): For $0 < T < \infty$ and $t \in [0,T]$,
$\Omega_\ell(t)$: liquid (melt) region,
$\Omega_s(t)$: solid (crystal) region,
$S(t)$: solid-liquid interface,
$\Omega(t) := \Omega_\ell(t) \cup \Omega_s(t) \cup S(t)$,
$\Gamma(t) := \partial\Omega(t)$,
$\nu = \nu(t,x)$: 3-dimensional unit vector normal to $\Gamma(t)$ at $x \in \Gamma(t)$,
$n = n(t,x)$: 3-dimensional unit vector normal to $S(t)$ at $x \in S(t)$,
$Q := \bigcup_{t\in(0,T)} \{t\} \times \Omega(t)$,
$\Sigma := \bigcup_{t\in(0,T)} \{t\} \times \Gamma(t)$,
$S := \bigcup_{t\in(0,T)} \{t\} \times S(t)$.

Next, we denote by $v_\Sigma := v_\Sigma(t,x)$ the normal speed of $\Gamma(t)$ at $(t,x) \in \Sigma$. With this v_Σ the 4-dimensional unit vector outward normal to Σ at each $(t,x) \in \Sigma$ is given by

$$\vec{\nu} := (\vec{\nu}_t, \vec{\nu}_x) = \frac{1}{(|v_\Sigma|^2 + 1)^{\frac{1}{2}}}(-v_\Sigma, \nu).$$

Similarly, with the normal speed $v_S := v_S(t,x)$ of $S(t)$ at $(t,x) \in S$, the 4-dimensional unit vector normal to S, pointing to the liquid region, is given by

$$\vec{n} := (\vec{n}_t, \vec{n}_x) = \frac{1}{(|v_S|^2 + 1)^{\frac{1}{2}}}(-v_S, n).$$

It is easily understood that by the crystal growth the shape of material domain $\Omega(t)$ changes with time and it yields a 3-dimensional convective vector field $\mathbf{v} := \mathbf{v}(t,x)$ in Q. The determination of $\mathbf{v}$ is also one of the important questions in the mathematical modeling of the Czochralski crystal growth process. It is reasonable to postulate that $\mathbf{v}$ is nothing but the pulling velocity v_p in the crystal and may be a solution of the incompressible Navier-Stokes (or simply Stokes) equation in the melt (see Crowley [1], DiBenedetto and O'Leary [3]). Nevertheless, in this paper, we assume that the convective field $\mathbf{v}$ is prescribed, too, satisfying that

$$\mathrm{div}\mathbf{v} = 0 \quad \text{in } \Omega(t),\ 0 < t < T, \tag{1.1}$$

$$\mathbf{v}\cdot\nu = v_\Sigma \quad \text{on } \Gamma(t),\ 0 < t < T. \tag{1.2}$$

Now, from the usual energy balance lows we derive the following system to determine the temperature field $\theta := \theta(t,x)$ and interface $S(t)$; note that $\theta(t,x)$ is the solution of a Stefan problem with prescribed convection $\mathbf{v}$ formulated in the non-cylindrical domain Q,

$$\text{(SPC)}\quad \begin{cases} \theta_t - c_\ell\Delta\theta + \mathbf{v}\cdot\nabla\theta = f \quad \text{in } Q_\ell := \displaystyle\bigcup_{t\in(0,T)} \{t\}\times\Omega_\ell(t), & (1.3)\\ \theta_t - c_s\Delta\theta + \mathbf{v}\cdot\nabla\theta = f \quad \text{in } Q_s := \displaystyle\bigcup_{t\in(0,T)} \{t\}\times\Omega_s(t), & (1.4)\\ \theta = 0, \quad \left(c_\ell\dfrac{\partial\theta}{\partial n} - c_s\dfrac{\partial\theta}{\partial n}\right) = L(\mathbf{v}\cdot n - v_S) \quad \text{on } S, & (1.5)\\ c_\ell\dfrac{\partial\theta}{\partial\nu} + n_0 c_\ell\theta = p \quad \text{on } \Sigma_\ell := \displaystyle\bigcup_{t\in(0,T)} \{t\}\times\{\partial\Omega_\ell(t)\setminus S(t)\}, & (1.6)\\ c_s\dfrac{\partial\theta}{\partial\nu} + n_0 c_s\theta = p \quad \text{on } \Sigma_s := \displaystyle\bigcup_{t\in(0,T)} \{t\}\times\{\partial\Omega_s(t)\setminus S(t)\}, & (1.7)\\ \theta(0,\cdot) = \theta_0 \quad \text{on } \Omega(0), \quad S(0) = S_0, & (1.8) \end{cases}$$

where we suppose that the phase change temperature is 0 for simplicity; c_ℓ, c_s and L are positive constants which are the heat conductivities and latent heat, respectively; f is a given heat source on Q, p is a boundary datum prescribed on Σ and n_0 is a positive constant; θ_0 is the initial temperature on $\Omega(0)$ and S_0 is the initial location of the solid-liquid interface, satisfying that

$$\theta_0 > 0 \quad \text{on } \Omega_\ell(0), \quad \theta_0 < 0 \quad \text{on } \Omega_s(0), \quad \theta_0 = 0 \quad \text{on } S_0. \tag{1.9}$$

As is well known, by using the enthalpy we reformulate this problem as a weak variational form. In this paper we prove its well-posedness.

2. Weak formulation

The enthalpy u is defined as follows:

$$u := \begin{cases} \theta + L & \text{if } \theta > 0, \\ [0, L] & \text{if } \theta = 0, \\ \theta & \text{if } \theta < 0. \end{cases}$$

Moreover we define a function $\beta : \mathbf{R} \to \mathbf{R}$ by

$$\beta(r) := \begin{cases} c_s r & \text{if } \ r < 0, \\ 0 & \text{if } \ 0 \le r \le L, \\ c_\ell (r - L) & \text{if } \ r > L. \end{cases}$$

Then β is a non-decreasing Lipschitz continuous function on $\mathbf{R}$, and its Lipschitz constant is $L_\beta := \max\{c_\ell, c_s\}$.

By using the enthalpy u our problem (SPC) is reformulated as an initial-boundary value problem for a degenerate parabolic equation of the following form

$$\text{(E)} \qquad \begin{cases} u_t - \Delta\beta(u) + \mathbf{v} \cdot \nabla u = f \quad \text{in } Q, \\ \dfrac{\partial \beta(u)}{\partial \nu} + n_0 \beta(u) = p \quad \text{on } \Sigma, \\ u(0) = u_0 \quad \text{on } \Omega(0), \end{cases}$$

where $u_0 := \theta_0 + L\chi_{\Omega_\ell(0)}$ with the characteristic function $\chi_{\Omega_\ell(0)}$ of $\Omega_\ell(0)$. In fact, multiply equations (1.3) and (1.4) by any test function $\eta \in C^2(\bar{Q})$ with $\eta = 0$ on $\Omega(T)$, and then integrate them over Q_ℓ and Q_s, respectively, and add these two resultants. Then, with the help of the Green-Stokes' formula and (1.1), (1.2), (1.5), (1.6) and (1.7) as well as the relations $d\Sigma = (|v_\Sigma|^2 + 1)^{\frac{1}{2}} d\Gamma(t)dt$ and $dS = (|v_S|^2 + 1)^{\frac{1}{2}} dS(t)dt$, we arrive at the following variational identity

$$\begin{aligned} &-\int_Q u\eta_t dxdt + \int_Q \nabla\beta(u) \cdot \nabla\eta dxdt - \int_\Sigma \frac{\partial\beta(u)}{\partial\nu} \eta d\Gamma(t)dt - \int_Q u(\mathbf{v} \cdot \nabla\eta) dxdt \\ &= \int_Q f\eta dxdt + \int_{\Omega(0)} u_0 \eta(0) dx \quad \text{for all } \eta \in C^2(\bar{Q}), \ \eta = 0 \text{ on } \Omega(T). \qquad (2.1) \end{aligned}$$

Next, in order to consider a weak formulation of the boundary conditions, for each $t \in [0, T]$, we take a harmonic function $g(t, \cdot)$ such that

$$\begin{cases} -\Delta g(t) = 0 \quad \text{in } \Omega(t), \\ \dfrac{\partial g(t)}{\partial \nu} + n_0 g(t) = p(t) \quad \text{on } \Gamma(t), \end{cases}$$

in fact, $g(t) \in H^1(\Omega(t))$ is a unique solution of the variational problem

$$\int_{\Omega(t)} \nabla g(t) \cdot \nabla\xi dx + n_0 \int_{\Gamma(t)} g(t)\xi d\Gamma(t) = \int_{\Gamma(t)} p(t)\xi d\Gamma(t) \quad \text{for all } \xi \in C^2(\overline{\Omega(t)}).$$

Also, we define the class W of test functions as follows:

$$W := \{w \in H^1(Q);\ w = 0 \quad \text{on } \Omega(T) \text{ (in the trace sense)}\}.$$

Then (2.1) can be rewritten in the form

$$-\int_Q uw_t dxdt + \int_Q \nabla(\beta(u) - g)\cdot\nabla w dxdt + \int_\Sigma n_0(\beta(u) - g)w d\Gamma(t)dt - \int_Q u(\mathbf{v}\cdot\nabla\eta)dxdt$$
$$= \int_Q fwdxdt + \int_{\Omega(0)} u_0 w(0,\cdot)dx \quad \text{for all } w \in W, \tag{2.2}$$

and as usual, this is regarded as a weak formulation of (E).

As to the solvability of two-phase Stefan problems without convection in cylindrical domains, the time-dependent subdifferential operator theory was skillfully applied by Damlamian [2]. The case of non-cylindrical domains was treated by Kenmochi and Pawlow [7] and only the existence result was there obtained, but the uniqueness question has been left open.

Now we formulate our main result. First of all we define the weak solution of our problem.

Definition 2.1 u is called a weak solution of (SPC) if $u \in L^2(Q)$, $\beta(u(t)) \in H^1(\Omega(t))$ for a.e. $t \in [0,T]$ with

$$\int_0^T |\beta(u)|^2_{H^1(\Omega(t))}dt < \infty,$$

$u(t,\cdot) \in L^2(\Omega(t))$ for all $t \in [0,T]$, the function

$$t \longmapsto \int_{\Omega(t)} u(t,x)\xi(x)dx \quad \text{is continuous on } [0,T] \text{ for each } \xi \in L^2_{\text{loc}}(\mathbf{R}^3),$$

and u satisfies the variational identity (2.2).

We suppose that the material domain $\Omega(t)$ depends smoothly on time t in the sense that there is a C^3-diffeomorphism $y = X(t,x)$ from $\bar{Q}$ onto $\bar{Q}_0$, with $Q_0 := (0,T)\times\Omega(0)$, satisfying properties

(1) $X(t,\cdot) := (X_1(t,x), X_2(t,x), X_3(t,x))$ maps $\overline{\Omega(t)}$ onto $\overline{\Omega(0)}$ for all $t \in [0,T]$;

(2) $X(0,\cdot) = I$ (identity) on $\overline{\Omega(0)}$.

We use the following notation:

$$\Omega_0 := \Omega(0),\ \Gamma_0 := \partial\Omega(0),\ \Sigma_0 := (0,T)\times\Gamma_0,\quad y = (y_1,y_2,y_3) \in \overline{\Omega_0};$$

and the inverse of $y = X(t,x)$ is denoted by $x = Y(t,y) := (Y_1(t,y), Y_2(t,y), Y_3(t,y))$.

Under some assumptions on the data $\mathbf{v}$, f, p and u_0, we prove:

Theorem 2.1 *Assume that* $f \in H^1(Q)$, $p \in C^1(\bar{\Sigma})$, $u_0 \in L^2(\Omega(0))$ *and* $\beta(u_0) \in H^1(\Omega(0))$. *Also, assume that* $\mathbf{v} \in C^1(\bar{Q})^3$ *and* (1.1)-(1.2) *are satisfied. Then there is one and only one weak solution u of* (SPC).

We give the sketch of the proof of Theorem 2.1 in the rest of this paper. For the detail proof see the forthcoming paper Fukao, Kenmochi and Pawlow [5].

3. Regular approximation for (SPC)

In this section, let us consider an approximate problem $(\mathrm{SPC})_\delta$, with parameter $\delta \in (0,1]$, for (SPC):

$$(\mathrm{SPC})_\delta \quad \begin{cases} u_{\delta,t} - \Delta\beta_\delta(u_\delta) + \mathbf{v}\cdot\nabla u_\delta = f_\delta \quad \text{in } Q, & (3.1)\\ \dfrac{\partial(\beta_\delta(u_\delta))}{\partial\nu} + n_0\beta_\delta(u_\delta) = p_\delta \quad \text{on } \Sigma, & (3.2)\\ u_\delta(0) = u_{0\delta} \quad \text{on } \Omega(0), & (3.3)\end{cases}$$

where β_δ, f_δ, p_δ and $u_{\delta,0}$ are regular approximations of β, f, p and u_0, respectively, as follows.

(1) β_δ is a smooth, increasing and Lipschitz continuous function on $\mathbf{R}$ such that

$$\delta \le \beta_\delta'(r)\left(= \frac{d}{dr}\beta_\delta(r)\right) \le C_0 \quad \text{for all } r \in \mathbf{R},$$

for a positive constant C_0, and such that

$$\beta_\delta \to \beta \quad \text{uniformly on } \mathbf{R} \text{ as } \delta \to 0;$$

we put $\hat\beta_\delta(r) := \int_0^r \beta_\delta(s)ds$ as well as $\hat\beta(r) := \int_0^r \beta(s)ds$ for all $r \in \mathbf{R}$.

(2) f_δ is a smooth function on $\bar Q$ such that

$$f_\delta \to f \quad \text{in } H^1(Q) \text{ as } \delta \to 0.$$

(3) p_δ is a smooth function on $\bar\Sigma$ such that

$$p_\delta \to p \quad \text{in } C^1(\bar\Sigma) \text{ as } \delta \to 0.$$

(4) $u_{0\delta}$ is a smooth function on $\overline{\Omega(0)}$ such that $u_{0\delta} \to u_0$ in $L^2(\Omega(0))$, $\beta_\delta(u_{0,\delta}) \to \beta(u_0)$ in $H^1(\Omega(0))$ as $\delta \to 0$ and the compatibility condition

$$\frac{\partial\beta_\delta(u_{0\delta})}{\partial\nu} + n_0\beta_\delta(u_{0\delta}) = p_\delta \quad \text{on } \overline{\Omega(0)} \tag{3.4}$$

holds.

We give first an existence-uniqueness result for the approximate problem $(\mathrm{SPC})_\delta$.

Lemma 3.1 *$(\mathrm{SPC})_\delta$ has one and only one solution u_δ such that u_δ and all the derivatives $u_{\delta,t}$, u_{δ,x_i}, u_{δ,x_ix_k} and u_{δ,tx_i}, $i,k = 1,2,3$, are Hölder continuous on $\bar Q$.*

Proof. By $y = X(t,x)$, we transform $(\mathrm{SPC})_\delta$ to a problem $(\overline{\mathrm{SPC}})_\delta$ formulated in the cylindrical domain Q_0:

$$(\overline{\mathrm{SPC}})_\delta \begin{cases} \bar{u}_{\delta,t} - \sum_{i,k=1}^{3} \dfrac{\partial}{\partial y_i}\left(a_{ik}\dfrac{\partial}{\partial y_k}\beta_\delta(\bar{u}_\delta)\right) + \mathbf{w}_1 \cdot \nabla\beta_\delta(\bar{u}_\delta) + \mathbf{w}_2 \cdot \nabla\bar{u}_\delta = \bar{f}_\delta \quad \text{in } Q_0, & (3.5) \\ \dfrac{\partial(\beta_\delta(\bar{u}_\delta))}{\partial\nu_A} + n_0\beta_\delta(\bar{u}_\delta) = \bar{p}_\delta \quad \text{on } \Sigma_0, & (3.6) \\ \bar{u}_\delta(0) = u_{0\delta} \quad \text{on } \Omega(0), & (3.7) \end{cases}$$

where $\bar{u}_\delta(t,y) := u_\delta(t, Y(t,y))$, $\bar{f}_\delta(t,y) := f_\delta(t, Y(t,y))$, $\bar{p}_\delta(t,y) := p_\delta(t, Y(t,y))$,

$$a_{ik}(t,y) := \sum_{j=1}^{3} \frac{\partial X_i}{\partial x_j}\frac{\partial X_k}{\partial x_j}, \quad i,k = 1,2,3,$$

$$\mathbf{w}_1 := (w_{11}, w_{12}, w_{13}) \quad \text{with } w_{1k} := \sum_{i,j=1}^{3} \frac{\partial}{\partial y_j}\left(\frac{\partial X_j}{\partial x_i}\right)\frac{\partial X_k}{\partial x_i}, \quad k = 1,2,3,$$

$$\mathbf{w}_2 := \frac{\partial X}{\partial t} + \mathbf{v}B \quad \text{with } B = \begin{pmatrix} \dfrac{\partial X_1}{\partial x_1} & \dfrac{\partial X_2}{\partial x_1} & \dfrac{\partial X_3}{\partial x_1} \\ \dfrac{\partial X_1}{\partial x_2} & \dfrac{\partial X_2}{\partial x_2} & \dfrac{\partial X_3}{\partial x_2} \\ \dfrac{\partial X_1}{\partial x_3} & \dfrac{\partial X_2}{\partial x_3} & \dfrac{\partial X_3}{\partial x_3} \end{pmatrix}$$

and

$$\frac{\partial(\cdot)}{\partial\nu_A} := \sum_{i,k=1}^{3} a_{ik}\frac{\partial(\cdot)}{\partial y_i}\bar{\nu}_k \quad \text{on } \Gamma_0,$$

where $\bar{\nu} = (\bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3)$ is the unit outward normal vector to Γ_0.

Since $X(0,\cdot) = I$ on $\bar{\Omega}_0$, the matrix $a_{ik}(0,y)$ is the unit on $\bar{\Omega}_0$ and hence $a_{ik}(t,y)$ is strictly positive definite on $\bar{\Omega}_0$ for $t \in [0,T']$ with a certain positive $T'(\leq T)$. Therefore $(\overline{\mathrm{SPC}})_\delta$ is (uniformly) parabolic quasi-linear equation with smooth coefficients on $Q_0(T') := (0,T') \times \Omega_0$, and by (3.4) the compatibility condition for initial and boundary data is satisfied. Now, apply the general existence-uniqueness theorem due to Ladyženskaja, Solonnikov and Ural'ceva [8;Chapter 5, section 7] to $(\overline{\mathrm{SPC}})_\delta$. Then we see that $(\overline{\mathrm{SPC}})_\delta$ has a unique solution $\bar{u}_\delta$ in the Hölder space $H^{2+\alpha,1+\alpha/2}(\overline{Q_0(T')})$ for a certain exponent $\alpha \in (0,1)$. It is also easy to check that $u_\delta(t,x) := \bar{u}_\delta(t, X(t,x))$ is a solution of $(\mathrm{SPC})_\delta$ on $Q(T') := \bigcup_{t\in(0,T')}\{t\} \times \Omega(t)$, satisfying the required regularities. If $T' < T$, then the solution u_δ can be extended beyond time T' by repeating the same argument as above with initial time T'. Finally we can construct a unique solution u_δ of $(\mathrm{SPC})_\delta$ on Q in the Hölder class. $\square$

Lemma 3.2 (Uniform estimate) *There exists a positive constant M_0, independent of parameter $\delta \in (0,1]$, such that*

$$\sup_{t\in[0,T]} |u_\delta(t)|^2_{L^2(\Omega(t))} + \sup_{t\in[0,T]} |\beta_\delta(u_\delta(t))|^2_{H^1(\Omega(t))}dt + \int_Q \left|\frac{\partial}{\partial t}\beta_\delta(u_\delta)\right|^2 dxdt \le M_0 \qquad (3.8)$$

$$\textit{for all } \delta \in (0,1].$$

Proof. We use essentially conditions (1.1) and (1.2) in order to get the uniformly estimates (3.8).

First, multiplying (3.1) by $\beta_\delta(u_\delta)$ and integrating over $Q(t) := \bigcup_{s\in(0,t)}\{s\} \times \Omega(s)$, we have by (1.1) and (1.2)

$$\int_{\Omega(t)} \hat{\beta}_\delta(u_\delta(t))dx + \int_{Q(t)} |\nabla\beta_\delta(u_\delta)|^2 dxds + n_0\int_0^t \int_{\Gamma(s)} |\beta_\delta(u_\delta)|^2 d\Gamma(s)ds$$

$$= \int_{Q(t)} f_\delta\beta_\delta(u_\delta)dxds + \int_0^t\int_{\Gamma(s)} p_\delta\beta_\delta(u_\delta)d\Gamma(s)ds + \int_{\Omega(0)} \hat{\beta}_\delta(u_{0,\delta})dx \qquad (3.9)$$

$$\text{for all } t\in[0,T].$$

From (3.9) we obtain a uniform estimate of the form

$$\sup_{t\in[0,T]} |u_\delta(t)|^2_{L^2(\Omega(t))} + \int_Q |\nabla\beta_\delta(u_\delta)|^2 dxdt \le M_1 \qquad (3.10)$$

for a positive constant M_1 independent of $\delta \in (0,1]$.

Next, just as (3.10), multiplying (3.1) by u_δ, we obtain a uniform estimate of the form

$$\int_Q \beta'_\delta(u_\delta)|\nabla u_\delta|^2 dxdt \le M_2 \qquad (3.11)$$

for a positive constant M_2, independent of $\delta \in (0,1]$.

The required estimate for $\partial\beta_\delta(u_\delta)/\partial t$ is obtained from that of the solution $\bar{u}_\delta$ of $(\overline{SPC})_\delta$. In fact, multiplying (3.5) by $\partial\beta_\delta(\bar{u}_\delta)/\partial t$ and integrating the resultant over $Q_0(t) := (0,t)\times\Omega_0$, we have

$$\int_{\Omega_0(t)} \beta'_\delta(\bar{u}_\delta)|\bar{u}_{\delta,s}|^2 dyds - \sum_{i,k=1}^3 \int_{Q_0(t)} \frac{\partial}{\partial y_k}\left(a_{ik}\frac{\partial\beta_\delta(\bar{u}_\delta)}{\partial y_i}\right)\frac{\partial\beta_\delta(\bar{u}_\delta)}{\partial s}dyds$$

$$+ \int_{Q_0(t)} (\mathbf{w}_1\cdot\nabla\beta_\delta(\bar{u}_\delta))\beta'_\delta(\bar{u}_\delta)u_{\delta,s}dyds + \int_{Q_0(t)} (\mathbf{w}_2\cdot\nabla\bar{u}_\delta)\beta'_\delta(\bar{u}_\delta)\bar{u}_{\delta,s}dyds \qquad (3.12)$$

$$= \int_0^t\int_{\Gamma_0} \bar{p}_\delta\frac{\partial\beta_\delta(\bar{u}_\delta)}{\partial ds}d\Gamma_0 ds + \int_{Q_0(t)} \bar{f}_\delta\frac{\partial\beta_\delta(\bar{u}_\delta)}{\partial s}dyds$$

$$\text{for all } t\in[0,T].$$

Here, for the time-dependent convex functional

$$\Phi_\delta(t;v) := \frac{1}{2}\sum_{i,k=1}^{3}\int_{\Omega_0} a_{ik}(t,y)\frac{\partial v}{\partial y_i}\frac{\partial v}{\partial y_k}dy + \frac{n_0}{2}\int_{\Gamma_0}|v|^2 d\Gamma_0 - \int_{\Gamma_0}\bar{p}_\delta(t,y)v d\Gamma_0$$

$$\text{for all } v \in H^1(\Omega_0)$$

we observe (cf. Kenmochi [5], Kenmochi and Pawlow [6]) that if $v \in W^{1,2}(0,T;L^2(\Omega_0)) \cap L^2(0,T;H^2(\Omega_0))$ and $v(0,\cdot) \in H^2(\Omega_0)$, then $\Phi_\delta(t,v(t))$ is absolutely continuous on $[0,T]$ and

$$\frac{d}{dt}\Phi_\delta(t,v(t)) + \sum_{i,k=1}^{3}\int_{\Omega_0} a_{ik}(t,y)\frac{\partial v(t,x)}{\partial y_i}\frac{\partial v(t,y)}{\partial t}dy \le R_0(\Phi_\delta(t,v(t)) + r_0) \tag{3.13}$$

$$\text{a.e. on } (0,T),$$

where R_0 and r_0 are positive constants independent of $\delta \in (0,1]$. Now, we take $\beta_\delta(\bar{u}_\delta)$ as a function v in (3.13) to obtain from (3.12) with the help of estimates (3.10) and (3.11) that

$$\sup_{t\in[0,T]}|\Phi_\delta(t,\beta_\delta(\bar{u}_\delta(t)))| + \int_Q \beta_\delta'(\bar{u}_\delta)|\bar{u}_{\delta,t}|^2 dydt \le M_3 \tag{3.14}$$

for a certain positive constant M_3 independent of $\delta \in (0,1]$. The estimates (3.10), (3.11) and (3.14) imply that (3.8) holds for some positive constant M_0 independent of $\delta \in (0,1]$. □

4. Proof of the theorem

Existence:

Let $\{u_\delta\}_{\delta\in(0,1]}$ be the family of approximate solutions of $(\text{SPC})_\delta$. By Lemma 3.2 with the standard compactness argument we can find a sequence $\{\delta_n\}$ with $\delta_n \to 0$ as $n \to +\infty$ and a function u such that

$$u_n := u_{\delta_n} \to u \quad \text{weakly in } L^2(Q),$$

$$\beta_{\delta_n}(u_n) \to \beta(u) \quad \text{in } L^2(Q) \text{ and weakly in } H^1(Q).$$

We now show that u is a weak solution of (SPC). To do so, multiply (3.1) by any test function $\eta \in C^2(\bar{Q})$ with $\eta(T,\cdot) = 0$ and integrate it over Q. Then we have by the Green-Stokes formula

$$-\int_Q u_n\eta_t dxdt - \int_\Sigma u_n\eta v_\Sigma d\Gamma(t)dt + \int_Q \nabla\beta_{\delta_n}(u_n)\cdot\nabla\eta dxdt + n_0\int_\Sigma \beta_{\delta_n}(u_n)\eta d\Gamma(t)dt$$

$$-\int_Q u_n(\mathbf{v}\cdot\nabla\eta)dxdt + \int_\Sigma u_n\eta(\mathbf{v}\cdot\nu)d\Gamma(t)dt$$

$$= \int_Q f_{\delta_n}\eta dxdt + \int_\Sigma p_{\delta_n}\eta d\Gamma(t)dt + \int_{\Omega(0)} u_{0\delta_n}\eta(0,\cdot)dx.$$

Here, noting condition (1.2) again and passing to the limit in n yield

$$-\int_Q u\eta_t dxdt + \int_Q \nabla\beta(u)\cdot\nabla\eta dxdt + n_0\int_\Sigma \beta(u)\eta d\Gamma(t)dt - \int_Q u(\mathbf{v}\cdot\nabla\eta)dxdt$$
$$= \int_Q f\eta dxdt + \int_\Sigma p\eta d\Gamma(t)dt + \int_{\Omega(0)} u_0\eta(0,\cdot)dx,$$

which is the required variational identity. Thus u is a weak solution of (SPC).

Uniqueness:

The idea of our uniqueness proof is due to Ladyženskaja, Solonnikov and Ural'ceva [8;Chapter 5,section 8], and this is also extensively used in Niezgódka and Pawlow [9], Rodrigues [11] and Rodrigues and Yi [12] for the uniqueness proof of generalized Stefan problems and continuous casting problems.

Let u_1 and u_2 be two weak solutions. Then

$$-\int_Q (u_1-u_2)\eta_t dxdt - \int_Q (\beta(u_1)-\beta(u_2))\Delta\eta dxdt + \int_\Sigma (\beta(u_1)-\beta(u_2))\frac{\partial\eta}{\partial\nu}d\Gamma(t)dt$$
$$+n_0\int_\Sigma(\beta(u_1)-\beta(u_2))\eta d\Gamma(t)dt - \int_Q (u_1-u_2)(\mathbf{v}\cdot\nabla\eta)dxdt = 0 \qquad (4.1)$$
$$\text{for all } \eta\in C^1(\bar{Q}) \text{ with } \eta(T,\cdot)=0.$$

As usual, consider the function

$$b(t,x) := \begin{cases} \dfrac{\beta(u_1(t,x))-\beta(u_2(t,x))}{u_1(t,x)-u_2(t,x)} & \text{if } u_1(t,x)\neq u_2(t,x), \\ 0 & \text{if } u_1(t,x)=u_2(t,x), \end{cases}$$

which is non-negative and bounded on Q. Then, by (4.1)

$$-\int_Q (u_1-u_2)\{\eta_t+b\Delta\eta+\mathbf{v}\cdot\nabla\eta\}dxdt + \int_\Sigma(\beta(u_1)-\beta(u_2))\left\{\frac{\partial\eta}{\partial\nu}+n_0\eta\right\}d\Gamma(t)dt = 0 \quad (4.2)$$
$$\text{for all } \eta\in C^1(\bar{Q}) \text{ with } \eta(T,\cdot)=0.$$

We now take a smooth and strictly positive approximation b_ε of b such that

$$b\le b_\varepsilon \quad \text{a.e. on } Q, \ \varepsilon\le b_\varepsilon\le C_1 \quad \text{a.e. on } Q$$
$$b_\varepsilon\to b \quad \text{a.e. on } Q \text{ as } \varepsilon\to 0,$$

where C_1 is a positive constant, and consider the following auxiliary linear parabolic equation $(\mathrm{P})_\varepsilon$ for any given $\ell\in\mathcal{D}(Q)$:

$$(\mathrm{P})_\varepsilon \quad \begin{cases} \eta_{\varepsilon,t}+b_\varepsilon\Delta\eta_\varepsilon+\mathbf{v}\cdot\nabla\eta_\varepsilon=\ell & \text{in } Q, \\ \dfrac{\partial\eta_\varepsilon}{\partial\nu}+n_0\eta_\varepsilon=0 & \text{on } \Sigma, \\ \eta_\varepsilon(T,\cdot)=0 & \text{on } \Omega(T). \end{cases} \qquad (4.3)$$

By the general theory of linear parabolic equations this problem has a unique solution $\eta_\varepsilon \in H^{2+\alpha,1+\alpha/2}(\bar{Q})$ and the following estimates are obtained:

$$\sup_{t\in[0,T]} |\eta_\varepsilon(t)|^2_{L^2(\Omega(t))} + \int_0^T |\nabla\eta_\varepsilon(t)|^2_{L^2(\Omega(t))}dt + \int_Q |b_\varepsilon||\Delta\eta_\varepsilon|^2 dxdt \le M_4, \tag{4.4}$$

where M_4 is a positive constant independent of $\varepsilon \in (0,1]$. In fact, (4.3) is obtained by multiplying (4.2) by $\Delta\eta_\varepsilon$. By (4.4), there exists a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ and a function $\eta \in L^2(Q)$ with $\nabla\eta \in L^2(Q)^3$ such that

$$\eta_{\varepsilon_n} \to \eta \quad \text{weakly in } L^2(Q),$$

$$\nabla\eta_{\varepsilon_n} \to \nabla\eta \quad \text{weakly in } L^2(Q)^3 \text{ as } n \to 0.$$

Taking η_{ε_n} as a test function η in (4.2) and passing to the limit in n, we see that

$$-\int_Q (u_1 - u_2)\ell dxdt = -\int_Q (u_1 - u_2)(b_{\varepsilon_n} - b)\Delta\eta_{\varepsilon_n} dxdt \to 0.$$

Therefore

$$\int_Q (u_1 - u_2)\ell dxdt = 0 \quad \text{for all } \ell \in \mathcal{D}(Q),$$

which implies that $u_1 = u_2$ a.e. on Q.

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